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Analytical solution on the non-linear vibration of a traveling wave ultrasonic motor

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Abstract A vibration model for the stator of traveling wave type ultrasonic motors has been presented using Kirchhoff plate theory. In this model, the vibration equation influenced by electric field is established based on the geometric equations, the constitutive equations and the balanced equations for the structure. This electric field distribution is assumed based on Maxwell equation. Shearing deformation, rotary inertia and damping of the piezoelectric ceramic are been taken into account. Firstly, the dynamics of the stator is discussed. The mode of the stator is derived and validated by comparing the resonant frequencies and modal shapes with those by finite element analysis. Then, the effect of the piezoelectric ceramic damping is also discussed. Finally, The friction layer is simplified as the axial spring. Using modal shapes as the weight function, the analytic solution under the forced vibration for the stator is computed by Galerkin method. The theoretical model and its analytic solution in this paper contribute to simplifying the model for the parametric study and understanding the vibration between the stator and the rotor.

Keywords Ultrasonic motor · Analytical solution · Kirchhoff plate · Non-linear vibration

1 Introduction

Ultrasonic motors use inverse piezoelectric effect to achieve the purpose of transform of electric energy and mechanical energy. Compared with the traditional electromagnetic motors, ultrasonic motors have many useful features such as high holding torque, high torque at low speed, quiet operation, simple structure, compactness in size and no electromagnetic interferences. They can be used in the fields of astronautics, medicine, the robot, and so on.

There are two methods of dynamical modeling of traveling wave type ultrasonic motors, the equivalent circuit modeling and the analytical modeling of structural dynamics. The analytical modeling includes finite element modeling and analytical modeling based on the plane theory. Equivalent circuit modeling can intuitionally reflect the running mechanism of ultrasonic motors. However, it can not depict the quantity of energy transport for ultrasonic motors. Finite element method can be used to establish the dynamic model of ultrasonic motors and it can effectively depict the quantity of energy transport. However, it can't get the analytic relationship between the elements of structures. Analytical modeling based on the plate theory can get the analytic solution of dynamic model and analytic expression for the elements of structures. So it can be used in the qualitative analysis of the dynamics of motors.

Analytical modeling based on the plane theory has been the subject of extensive research all over world. Takano et al. [1], Hagedorn and Wallaschek [2], Yang and Que [3], Friend and Stutts [4] used the annular plate theory to research the dynamics of traveling wave type ultrasonic motors. In their studies, shearing deformation, rotary inertia was neglected and piezoelectric-coupled effect and the laminated nature of the stator have not been modeled completely. Hagood and McFarland [5] used the laminated annular thin plate with clamped boundary condition at the inner edge and free boundary condition at the outer edge to establish the dynamic model of stator. They assumed that the distribution of electric potential is uniform in the

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thickness direction, which violate static electricity Maxwell Equation. In above modeling, just Friend and Stutts studied the effect of the structural damping.

In the present work, a model of the vibration for the stator of traveling wave type ultrasonic motors have been presented using Kirchhoff plate theory. In this model, the vibration equation influenced by electric field is established based on geometric equations, constitutive equations and balanced equations for structures. The electric field contents Maxwell equation. Shearing deformation, rotary inertia and damping of the piezoelectric ceramic are been taken into account. The concept of the mode of coupling vibration is put forward and the formula used to calculate its frequency of natural vibration is presented. The effect of friction layer is discussed and the friction layer is simplified as the axial spring. Galerkin method is used to calculate the analytical solution of the vibration of stators.

2 Structural mechanical model of stator

2.1 Physical model

A stator consists of one host layer and one piezoelectric layer. The host layer is a metallic plate. There are teeth on the host layer. Taking no account of teeth, the stator is considered as a laminated annular plate of constant thickness showed in Fig. 1.

It supposed that the thickness of the host layer is $h_{11}+h_{12}$, the thickness of the piezoelectric layer is h_2, the inner radius of annular plate is r_1, the outer radius is r_2. As an elastic structure, there is a neutral layer in the host layer, which carves up the host to two sections. Their thicknesses are h_{11} and h_{12} respectively. The displacements of particles on the neutral layer are only in the transverse direction. The inner boundary of annular plate is clamped edge, and the outer boundary is free edge. By Kirchhoff plate theory, the displacements in radial, tangential and transverse direction in stator can be expressed as

$$\begin{aligned} u_r &= -z\frac{\partial u_z}{\partial r} \\ u_\theta &= -z\frac{\partial u_z}{r\partial \theta} \\ u_z &= w(r,\theta,t) \end{aligned} \tag{1}$$

In which $w(r,\theta,t)$ is the displacement of the particles in transverse direction on neutral layer.

2.2 Geometrical equations

The kinematical fields in the host plate and piezoelectric layer are given by

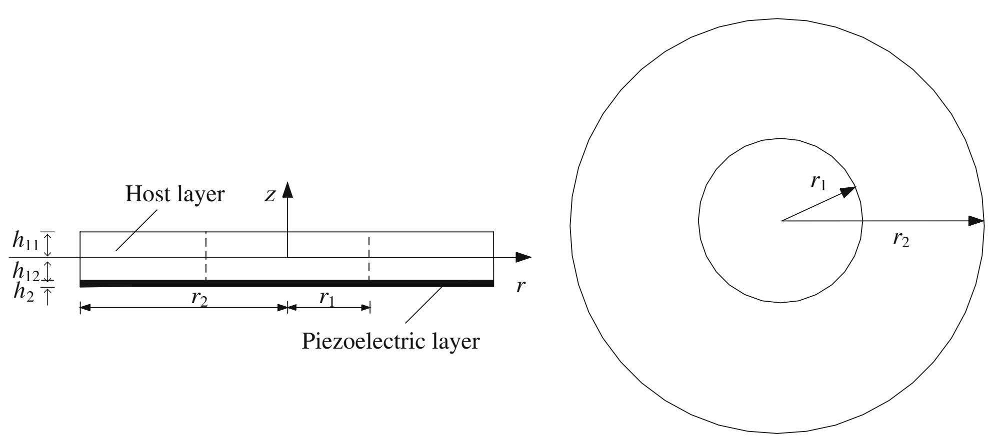
$$\begin{aligned} \varepsilon_r &= -z\frac{\partial^2 u_z}{\partial r^2} \\ \varepsilon_\theta &= -z\left(\frac{\partial u_z}{r\partial r}+\frac{\partial^2 u_z}{r^2\partial\theta^2}\right) \\ \gamma_{r\theta} &= -2z\left(\frac{\partial^2 u_z}{r\partial r\partial\theta}-\frac{\partial u_z}{r^2\partial\theta}\right) \end{aligned} \tag{2}$$

2.3 Constitutive relations in the host plate

A stator consists of host and piezoelectric ceramic. There is electric field in piezoelectric ceramic. Vibration for the host will be created by electric field. Hence it contents the constitutive relations and they are expressed as

$$\begin{aligned} \sigma_r^S &= \frac{E}{1-\mu^2}(\varepsilon_r+\mu\varepsilon_\theta) \\ \sigma_\theta^S &= \frac{E}{1-\mu^2}(\varepsilon_\theta+\mu\varepsilon_r) \\ \tau_{r\theta}^S &= \frac{E}{2(1+\mu)}\gamma_{r\theta} \end{aligned} \tag{3}$$

Fig. 1 Simple view of a stator



(a) Front Elevation **(b)** Plan

In which superscript S represents the variable in the host, E and μ are Young's modulus and Poisson ratio of the material.

From Eq. 3, we can get

$$\begin{aligned} \sigma_r^S &= -\frac{zE}{1-\mu^2}\left(\frac{\partial^2 u_z}{\partial r^2}+\mu\frac{\partial u_z}{r\partial r}+\mu\frac{\partial^2 u_z}{r^2\partial\theta^2}\right)\\ \sigma_\theta^S &= -\frac{zE}{1-\mu^2}\left(\mu\frac{\partial^2 u_z}{\partial r^2}+\frac{\partial u_z}{r\partial r}+\frac{\partial^2 u_z}{r^2\partial\theta^2}\right)\\ \tau_{r\theta}^S &= -\frac{zE}{1+\mu}\left(\frac{\partial^2 u_z}{r\partial r\partial\theta}-\frac{\partial u_z}{r^2\partial\theta}\right) \end{aligned} \tag{4}$$

2.4 Balanced equations

According as Kirchhoff plate theory, the strain in transverse direction is ignored, but the shear force must be thought over. Otherwise, without taking account of the damping of the host, it assumed that the damping of the piezoelectric layer is proportion to the vibration velocity, and rate coefficient of the damping is η_1. The balanced equations in the host are expressed as

$$\begin{aligned} &\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}+\frac{\partial\tau_{zr}}{\partial z}-\eta\frac{\partial u_r}{\partial t}=0\\ &\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{r\theta}}{\partial r}+\frac{2\tau_{r\theta}}{r}+\frac{\partial\tau_{z\theta}}{\partial z}-\eta\frac{\partial u_\theta}{\partial t}=0\\ &\frac{\partial\tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\tau_{rz}}{r}+\frac{\partial\sigma_z}{\partial z}+\rho\frac{\partial^2 u_z}{\partial t^2}=0 \end{aligned} \tag{5}$$

From Eq. 5, we can get

$$\begin{aligned} &\frac{\partial M_r}{\partial r}+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}+\frac{M_r-M_\theta}{r}-\eta_1\int_{-h-h_1}^{-h} z\frac{\partial u_z}{\partial t}dz=Q_r\\ &\frac{1}{r}\frac{\partial M_\theta}{\partial\theta}+\frac{\partial M_{r\theta}}{\partial r}+\frac{2M_{r\theta}}{r}-\eta_1\int_{-h-h_1}^{-h} z\frac{\partial u_\theta}{\partial t}dz=Q_\theta+\frac{h_{11}\partial P_\theta}{r\partial\theta}-P_\theta\\ &\frac{\partial Q_r}{\partial r}+\frac{1}{r}\frac{\partial Q_\theta}{\partial\theta}+\frac{Q_r}{r}+\rho\int_{-h-h_1}^{h}\frac{\partial^2 u_z}{\partial t^2}dz=0 \end{aligned} \tag{6}$$

In which when the position (r,θ,z) is in the host, η=0; when the position (r,θ,z) is in the piezoelectric layer, $\rho=\rho_1$, $\eta=\eta_1$.

Considering the effect of the friction layer, the friction layer is simplified as the axial spring. Its coefficient of elasticity is K_z. The domain of friction layer is annular field ($r_3 \le r \le r_2$).In this domain, the stator stands the axial pressure $P_z(r,t)$ and circumferential shear force $P_\theta(r,t)$. They can be expressed as

$$P_z(r,t)=\begin{cases} K_z(w-\delta) & w-\delta>0\\ 0 & w-\delta\le 0\end{cases} \tag{7}$$

$$P_\theta(r,t)=\mathrm{sign}(\dot{u}_\theta-v)f_\theta P_z(r,t) \tag{8}$$

In which δ is the distance between the free interfaces of the stator and the rotor. When the free interfaces of the rotor on the upward side of the stator's, δ is a positive value. On the opposition, it is a negative value. f_θ is the coefficient of friction between the stator and the rotor.

3 Vibration and electricity coupling equation

3.1 Piezoelectric equations

Because of the piezoelectric ceramic is polarized in z-direction, it is considered that electric displacements and electric field intensities are non-zero in z direction only. It yields $\boldsymbol{D}=(0 \quad 0 \quad D_z)^T$ and $\boldsymbol{E}=(0 \quad 0 \quad E_z)^T$. The second kind of piezoelectric equations can be simplified as

$$\begin{aligned} D_z &= \Xi_z E_z+e_{31}\varepsilon_r+e_{31}\varepsilon_\theta\\ \sigma_r^E &= -e_{31}E_z+C_{11}^E\varepsilon_r+C_{12}^E\varepsilon_\theta\\ \sigma_\theta^E &= -e_{31}E_z+C_{12}^E\varepsilon_r+C_{22}^E\varepsilon_\theta\\ \tau_{r\theta}^E &= 0.5(C_{11}^E-C_{12}^E)\gamma_{r\theta} \end{aligned} \tag{9}$$

In which C_{11}^E, C_{12}^E, C_{22}^E are the reduce modules of elasticity, Ξ_z, e_{31} are the reduce dielectric coefficients, E_z, D_z are the electric field intensities and the electric displacement in z direction.

$$\begin{aligned} D_z &= \Xi_z E_z - z\left(e_{31}\frac{\partial^2 u_z}{\partial r^2}+e_{31}\frac{\partial u_z}{r\partial r}+e_{31}\frac{\partial^2 u_z}{r^2\partial\theta^2}\right)\\ \sigma_r^E &= -e_{31}E_z - z\left(C_{11}^E\frac{\partial^2 u_z}{\partial r^2}+C_{12}^E\frac{\partial u_z}{r\partial r}+C_{12}^E\frac{\partial^2 u_z}{r^2\partial\theta^2}\right)\\ \sigma_\theta^E &= -e_{31}E_z - z\left(C_{12}^E\frac{\partial^2 u_z}{\partial r^2}+C_{22}^E\frac{\partial u_z}{r\partial r}+C_{22}^E\frac{\partial^2 u_z}{r^2\partial\theta^2}\right)\\ \tau_{r\theta}^E &= -z\left(C_{11}^E-C_{12}^E\right)\left(\frac{\partial^2 u_z}{r\partial r\partial\theta}-\frac{\partial u_z}{r^2\partial\theta}\right) \end{aligned} \tag{10}$$

Otherwise, the electric field intensities $\boldsymbol{E}$ and the electric displacement $\boldsymbol{D}$ must contents Maxwell equation and it has

$$\begin{aligned} &\mathrm{rot}\boldsymbol{E}=0\\ &\mathrm{div}\boldsymbol{D}=0 \end{aligned} \tag{11}$$

3.2 Electric field intensities of the piezoelectric layer

The electric field intensities are defined as the gradient of the electromotive force V and it is expressed as

$$\boldsymbol{E}=-\mathrm{grad}V(r,\theta,z,t) \tag{12}$$

which contents Eq. 9. Because the electric displacement is non-zero in z direction only, the Maxwell equation can be expressed as

$$\frac{\partial D_z}{\partial z}=0 \tag{13}$$

Substituting Eq. 10 into Eq. 13, it gives

$$\frac{\mathrm{d}E_z(r,\theta,z,t)}{\mathrm{d}z}=\alpha_E(r,\theta,t) \tag{14}$$

In which $\alpha_E(r,\theta,t)=\frac{e_{31}}{\Xi_z}\Delta w(r,\theta,t)$.

Integrating Eq. 14 about variable z, it yields

$$E_z(r,\theta,z,t) = E_0(r,\theta,t) + \alpha_E(r,\theta,t)(z + h_{12} + 0.5h_2) \tag{15}$$

By Eqs. 12 and 15, it yields the equation of the electromotive force V as

$$V = -\int E_z dz = -\frac{1}{2}\alpha_E(z+h_{12})(z+h_{12}+h_2) - E_0(z+h_{12}) + C \tag{16}$$

In which C is an independent constant with variable z.

3.3 Piezoelectric polarization

The electromotive force at the two piezoelectric poles is correlative with its polarization. Figure 2 expresses as the manner of the piezoelectric polarization of the Φ60 traveling wave type ultrasonic motor [6].

Figure 2 shows that there are two phases potential $V_A(t)$, $V_B(t)$ on piezoelectric ceramic. The symbolic function of the electromotive force on free surface of piezoelectric ceramic is defined as

$$\varphi_A(\theta) = \begin{cases} 1 & \theta \in \texttt{positive.phase.A} \\ 0 & \theta \in \texttt{phase.A} \\ -1 & \theta \in \texttt{positive.phase.A} \end{cases} \tag{17}$$

$$\varphi_B(\theta) = \begin{cases} 1 & \theta \in \texttt{positive.phase.B} \\ 0 & \theta \in \texttt{phase.B} \\ -1 & \theta \in \texttt{positive.phase.B} \end{cases} \tag{18}$$

The potential function on the piezoelectric ceramic is expressed as

$$V(\theta,t) = \boldsymbol{\Phi}^T(\theta)\boldsymbol{V}(t) \tag{19}$$

In which $\boldsymbol{\Phi}(\theta) = (\varphi_A(\theta)\ \varphi_B(\theta))^T$; $\boldsymbol{V}(t) = (V_A(t)\ V_B(t))^T$.

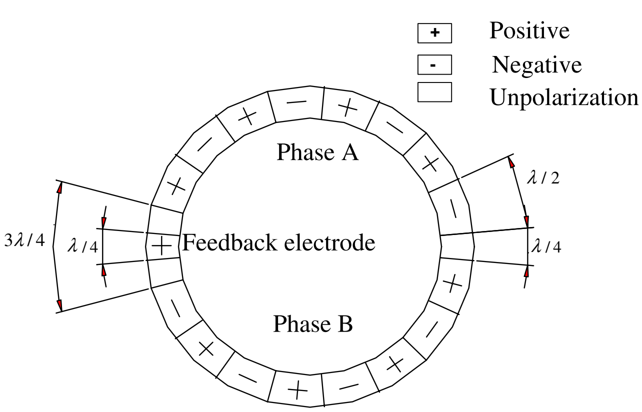


Fig. 2 The manner of the piezoelectric polarization

Boundary condition is

1. Adhesive surface

$$V(r,\theta,-h_{12},t) = 0 \tag{20}$$

2. Non-adhesive surface

$$V(r,\theta,-(h_{12}+h_2),t) = \boldsymbol{\Phi}^T\boldsymbol{V}(t) \tag{21}$$

Substituting Eq. 16 into Eqs. 20 and 21, it obtains

$$E_0 = \frac{1}{h_2}\boldsymbol{\Phi}^T\boldsymbol{V}(t), C = 0 \tag{22}$$

Substituting the equation into Eq. 16, it has

$$V = -0.5\frac{e_{31}}{\Xi_z}\nabla^2 w(r,\theta,t)(z+h_{12})(z+h_{12}+h_2) - \frac{z+h_{12}}{h_2}\boldsymbol{\Phi}^T\boldsymbol{V}(t) \tag{23}$$

4 Vibration equations of the stator

4.1 Modal analysis of the stator

It is supposed that M_r, M_θ, $M_{r\theta}$ are the bending and twisting moments, Q_r, Q_θ are the transverse shearing forces. They are defined as

$$\begin{aligned}(M_r, M_\theta, M_{r\theta}) &= \int_{-h_{12}-h_2}^{h_{11}} (\sigma_r, \sigma_\theta, \sigma_{r\theta})\mathrm{z}\mathrm{d}z \\ (Q_r, Q_\theta) &= \int_{-h_{12}-h_2}^{h_{11}} (\tau_{rz}, \tau_{\theta z})\mathrm{d}z\end{aligned} \tag{24}$$

From Eqs. 24 and 6, it obtains

$$\begin{aligned}
M_r &= -\left[\left(d\Delta w - 2A_1\left(\frac{\partial w}{r\partial r} + \frac{\partial^2 w}{r^2\partial\theta^2}\right) + \boldsymbol{A}_2\boldsymbol{V}(t)\right]\\
M_\theta &= -\left[\left(d\Delta w - 2A_1\frac{\partial^2 w}{\partial r^2} + \boldsymbol{A}_2\boldsymbol{V}(t)\right]\\
M_{r\theta} &= -2A_1\left(\frac{\partial^2 w}{r\partial r\partial\theta} - \frac{\partial w}{r^2\partial\theta}\right)\\
Q_r &= -d\frac{\partial}{\partial r}\Delta w + \varsigma\frac{\partial^2 w}{\partial r\partial t}\\
Q_\theta &= -\frac{1}{r}\left[\left(d\frac{\partial}{\partial\theta}\Delta w + \frac{\partial}{r\partial\theta}\boldsymbol{A}_2\boldsymbol{V}(t)\right] + \varsigma\frac{\partial^2 w}{r\partial\theta\partial t}
\end{aligned} \tag{25}$$

In which $d = \frac{E}{1-\mu^2}\int_{-h_{12}}^{h_{11}} z^2\mathrm{d}z + C_{11}^E\int_{-h_{12}-h_2}^{-h_{12}} z^2\mathrm{d}z + \frac{e_{31}^2}{\Xi_z}\int_{-h_{12}-h_2}^{-h_{12}} \mathrm{z}(\mathrm{z}+h_{12}+0.5h_2)\mathrm{d}z$,

$$A_1 = \frac{0.5E}{1+\mu}\int_{-h_{12}}^{h_{11}} z^2\mathrm{d}z + 0.5\left(C_{11}^E - C_{12}^E\right)\int_{-h_{12}-h_2}^{-h_{12}} z^2\mathrm{d}z,$$

$$\boldsymbol{A}_2 = -e_{31}(h_{12}+0.5h_2)\boldsymbol{\Phi}^T, \varsigma = \frac{1}{3}\eta_1\left((h_{12}+h_2)^3 - h_{12}^3\right)\circ$$

The vibration equation is expressed as

$$\frac{\partial Q_r}{\partial r} + \frac{\partial Q_\theta}{r\partial\theta} + \frac{Q_r}{r} = \int_{-h}^{h}\rho_1\frac{\partial^2 u_z}{\partial t^2}\mathrm{d}z + \int_{-h-h_1}^{-h}\rho_2\frac{\partial^2 u_z}{\partial t^2}\mathrm{d}z \tag{26}$$

Substituting Eq. 25 into the equation mentioned above, it has

$$d\Delta^2 w - \varsigma\Delta\dot{w} + A_3\ddot{w} + p\left(w, \dot{w}\right) = -\frac{\partial^2}{r^2\partial\theta^2}\boldsymbol{A}_2\boldsymbol{V}(t) \tag{27}$$

In which $p\left(w, \dot{w}\right) = -\frac{\partial P_\theta}{r\partial\theta} + h_{11}\frac{\partial^2 P_\theta}{r^2\partial\theta^2}$, $A_3 = (h_{11} + h_{12})\rho_1 + h_2\rho_2$.

Equation 27 expresses the vibration equation affected by electric field for the stator, where $-r^{-2}\left(\partial^2/\partial\theta^2\right)\boldsymbol{A}_2\boldsymbol{V}(t)$ is the force of the electric field, $-\eta\Delta w$ shows the effect of the damping of piezoelectric ceramic, $d\Delta^2 w$ shows the mutual effect on stator vibration and electric field. $p\left(w, \dot{w}\right)$ shows the non-linear coupling effect between the stator and friction layer on the rotor.

The corresponding boundary conditions are

$$\begin{aligned} w(r_1, \theta, t) = \frac{\partial w(r_1, \theta, t)}{\partial r} = 0 \\ M_r(r_2, \theta, t) = Q_r(r_2, \theta, t) = 0 \end{aligned} \tag{28}$$

Considering the free vibration system of stators, let $\boldsymbol{A}_2\boldsymbol{V}(t)$=0 and $p\left(w, \dot{w}\right) = 0$. Then the corresponding vibration equation is expressed as

$$d\Delta^2 w - \varsigma\Delta\dot{w} + A_3\ddot{w} = 0 \tag{29}$$

Suppose the solution of Eq. 29 is expressed as

$$w(r, \theta, t) = W(r)\sin(p\theta)e^{(-\xi\omega + \mathrm{i}\omega)t} \tag{30}$$

In which ξ is damping ratio, ω is vibration frequency, p is wave number.

Substituting Eq. 30 into Eq. 29, it obtains

$$\begin{aligned} d\Delta^2 W - A_3\left(\xi^2 + 1\right)\omega^2 W = 0 \\ \varsigma\Delta W + 2\xi\omega A_3 W = 0 \end{aligned} \tag{31}$$

In which $\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} - \frac{p^2}{r^2}$.

Define $\lambda^4 = \frac{(\xi^2+1)\omega^2 A_3}{d}$, according to Eq. 31, it obtains

$$\left(\Delta W - \lambda^2\right)\left(\Delta W + \lambda^2\right) = 0 \tag{32}$$

According to Eq. 32, it obtains the generic solution of Eq. 29

$$w = \left[c_1 J_p(\lambda r) + c_2 I_p(\lambda r) + c_3 Y_p(\lambda r) + c_4 K_p(\lambda r)\right] \sin(p\theta)e^{(-\xi + \mathrm{i})\omega t} \tag{33}$$

In which J and Y are first and secondary kinds of Bessel functions respectively, I and K are first and secondary kinds of modified Bessel functions respectively.

Substituting Eq. 33 into Eq. 28, it obtains

$$\mathfrak{A}\boldsymbol{c} = \boldsymbol{0} \tag{34}$$

In which

$$\mathfrak{A} = \begin{pmatrix} J_p(\lambda r)|_{r=r_1} & I_p(\lambda r)|_{r=r_1} & Y_p(\lambda r)|_{r=r_1} & K_p(\lambda r)|_{r=r_1} \\ \frac{\partial}{\partial r}J_p(\lambda r)|_{r=r_1} & \frac{\partial}{\partial r}I_p(\lambda r)|_{r=r_1} & \frac{\partial}{\partial r}Y_p(\lambda r)|_{r=r_1} & \frac{\partial}{\partial r}K_p(\lambda r)|_{r=r_1} \\ \boldsymbol{Q}_r\left(J_p(\lambda r)\right)|_{r=r_2} & \boldsymbol{Q}_r\left(I_p(\lambda r)\right)|_{r=r_2} & \boldsymbol{Q}_r\left(Y_p(\lambda r)\right)|_{r=r_2} & \boldsymbol{Q}_r\left(K_p(\lambda r)\right)|_{r=r_2} \\ \boldsymbol{M}_r\left(J_p(\lambda r)\right)|_{r=r_2} & \boldsymbol{M}_r\left(I_p(\lambda r)\right)|_{r=r_2} & \boldsymbol{M}_r\left(Y_p(\lambda r)\right)|_{r=r_2} & \boldsymbol{M}_r\left(K_p(\lambda r)\right)|_{r=r_2} \end{pmatrix};$$

$$\boldsymbol{c} = (c_1 \quad c_2 \quad c_3 \quad c_4)^T;\ \boldsymbol{M}_r = -\left[d\Delta - 2A_1\left(\frac{\partial}{r\partial r} - \frac{p^2}{r^2}\right)\right];$$

$\boldsymbol{Q}_r = -d\frac{\partial}{\partial r}\Delta + \varsigma(-\xi + \mathrm{i})\omega\frac{\partial}{\partial r}$, which are differential operator.

If the Eq. 34 has a nonzero solution, the determinant of the corresponding coefficient matrix must be zero. Define the determinant of the coefficient matrix of Eq. 34 to be zero and it obtains

$$\det(\mathfrak{A}) = 0 \tag{35}$$

From Eqs. 35 and 31, the natural frequency ω and damping coefficient ξ of stators can be obtained. Then from Eq. 34, modal shapes can be obtained, which the calculated mode can reflect effect of electric field.

4.2 Analytic solution of vibration of the stator

The corresponding modal shapes to the mode are $\{W(r)\cos(p\theta), W\sin(p\theta)\}$, because the motor operates under the $\mathbf{B}_{09}$ mode. Under the mode, the solution of Eq. 27 is

$$w = W(r)(a\cos(p\theta - \omega t) + b\sin(p\theta - \omega t)) \tag{36}$$

It satisfies the boundary condition 28.The Garlerkin method is used to solve Eq. 27. Substitute Eq. 36 into Eq. 27 and consider the form of weak solution of Eq. 27. We obtain

$$\begin{aligned} &\frac{1}{T}\int_0^T\int_0^{2\pi}\int_{r_1}^{r_2}\left(d\Delta^2 w - \varsigma\Delta\dot{w} + A_3\ddot{w} + p\left(w, \dot{w}\right)\right)W\cos(p\theta - \omega t)r\mathrm{d}r\mathrm{d}\theta\mathrm{d}t \\ &\quad = -\frac{1}{T}\int_0^T\int_0^{2\pi}\int_{r_1}^{r_2}\frac{\partial^2}{r^2\partial\theta^2}\boldsymbol{A}_2\boldsymbol{V}(t)W\cos(p\theta - \omega t)r\mathrm{d}r\mathrm{d}\theta\mathrm{d}t \\ &\frac{1}{T}\int_0^T\int_0^{2\pi}\int_{r_1}^{r_2}\left(d\Delta^2 w - \varsigma\Delta\dot{w} + A_3\ddot{w} + p\left(w, \dot{w}\right)\right)W\sin(p\theta - \omega t)r\mathrm{d}r\mathrm{d}\theta\mathrm{d}t \\ &\quad = -\frac{1}{T}\int_0^T\int_0^{2\pi}\int_{r_1}^{r_2}\frac{\partial^2}{r^2\partial\theta^2}\boldsymbol{A}_2\boldsymbol{V}(t)W\sin(p\theta - \omega t)r\mathrm{d}r\mathrm{d}\theta\mathrm{d}t \end{aligned} \tag{37}$$

Table 1 Physical parameters of the stator.

Attribute	Host	Piezoelectric ceramic
Young's modulus (N/m^2)	E=113×10^9	$C_{11}^E = 85.7 \times 10^9$ $C_{12}^E = 24.7 \times 10^9$ $C_{22}^E = 85.7 \times 10^9$ Ξ_z=5.84×10^{-9} e_{31}=−20
Poisson ratio	μ=0.28	μ=0.22
Density (kg/m^3)	ρ=8760	ρ=7600

In which T is periodic.

Equation 37 is a algebraic equation including parameter a and b. Though solving a and b, solution of weighted residual of Eq. 27 can be obtained. Because of $p(w, \dot{w})$, the equation above is not a linear algebraic equation. It is difficult to solve it directly. In this paper, the model is equal to an optimization problem.

Presetting the parameters a* and b*, then $w = W(r)(a^* \cos(p\theta - \omega t) + b^* \sin(p\theta - \omega t))$ is a function about r,θ,t. To the settled r,t, in one periodic 2π of parameter θ, the interval (0,2π) can be comparted into three parts, I_1, I_2 and I_3, in which $w-\delta>0$ and $\dot{u_\theta} - v \geq 0$ in I_1; $w-\delta>0$ and $\dot{u_\theta} - v < 0$ in I_2; $w-\delta\leq 0$ in I_3.

Substitute these into Eq. 37, we can get

$$-Ta - \left(2\pi\xi\omega^2 A_3 \int_{r_1}^{r_2} W^2 \mathrm{d}x + S\right) b = \int_{r_1}^{r_2} W \frac{1}{2r} (f_1(p) + f_2(p)) \mathrm{d}r$$
$$\left(2\pi\xi\omega^2 A_3 \int_{r_1}^{r_2} W^2 \mathrm{d}x + S\right) a - Tb = \int_{r_1}^{r_2} W \frac{1}{2r} (f_1(p) - f_2(p)) \mathrm{d}r \tag{38}$$

In which $S = \frac{1}{2}(|I_1| - |I_2|) \int_{r_3}^{r_2} f_\theta K_z p W^2 dr; T = \frac{1}{2}(|I_1| - |I_2|) \int_{r_3}^{r_2} f_\theta K_z h_{11} \frac{p^2}{r} W^2 dr$

$$f_1(p_i) = -0.5 e_{31}(h_{12} + 0.5h_2)$$
$$Vp\left(-0.5(\sin(p\theta_1) + \sin(p\theta_9)) + \sum_{j=2}^{8} (-1)^j \sin(p\theta_j)\right)$$
$$f_2(p) = 0.5 e_{31}(h_{12} + 0.5h_2)$$
$$Vp\left(-0.5(\cos(p\theta_1) + \cos(p\theta_9)) + \sum_{j=2}^{8} (-1)^j \cos(p\theta_j)\right) \tag{39}$$

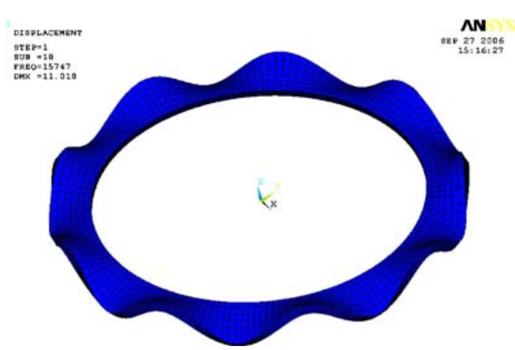


Fig. 3 The mode B$_{09}$ of the stator calculated by finite element method

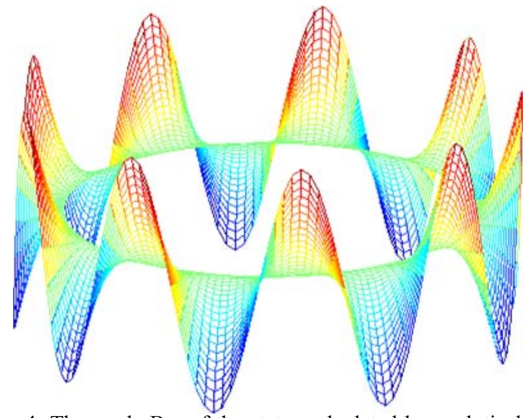


Fig. 4 The mode B$_{09}$ of the stator calculated by analytical method

In which $\theta_j = \frac{\pi}{36} + (j-1)\frac{\pi}{9}, j = 1, 2, \ldots, 9$.

5 Numerical simulation

5.1 Modal analysis

In order to validate the algorithm of this paper, an annular stator without teeth is discussed (Table 1). The external diameter of the stator is 56 mm. The inner diameter is 44 mm. The thickness of the elastic structure is 1.5 mm. The thickness of the piezoelectric ceramic is 0.5 mm. The frequency and mode of vibration in the condition with no damping are calculated by finite element method (Fig. 3) and the method given by this paper (Fig. 4) respectively.

Table 2 shows the frequency of vibration of stator, as the numbers of wave are 7, 8, 9 respectively.

Output efficiency of motors is low. One of the reasons is the heat generated from the piezoelectric ceramic. The heat of the piezoelectric ceramic is caused by high damping. Hence, considering the effect of damping is an essential to establish effective model of carrying energy. In this paper, a vibration equation considering the effect of damping is established. It is also used to discuss the damping's effect on vibration frequency.

In fact, firstly damping coefficient ς of the material is certain. The damping ratio ξ and frequency ω can be

Table 2 Vibration frequency kHz.

Wave number p	Finite element method	This method	Error (%)
p=7	12.894	12.225	−5.19
p=8	14.194	13.604	−4.16
p=9	15.747	15.716	−0.20

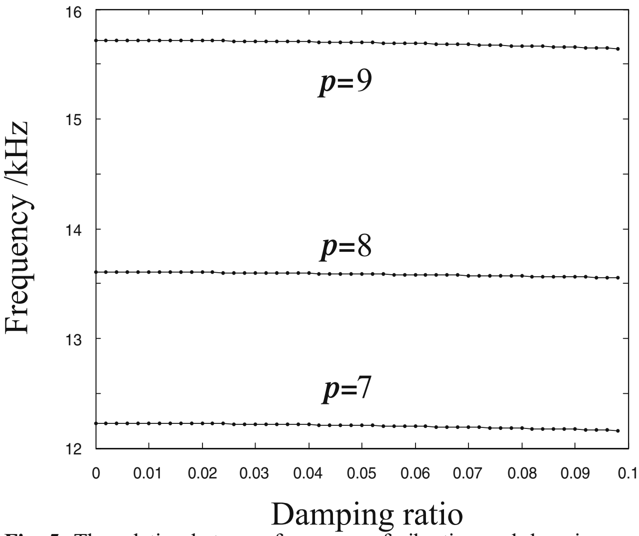


Fig. 5 The relation between frequency of vibration and damping

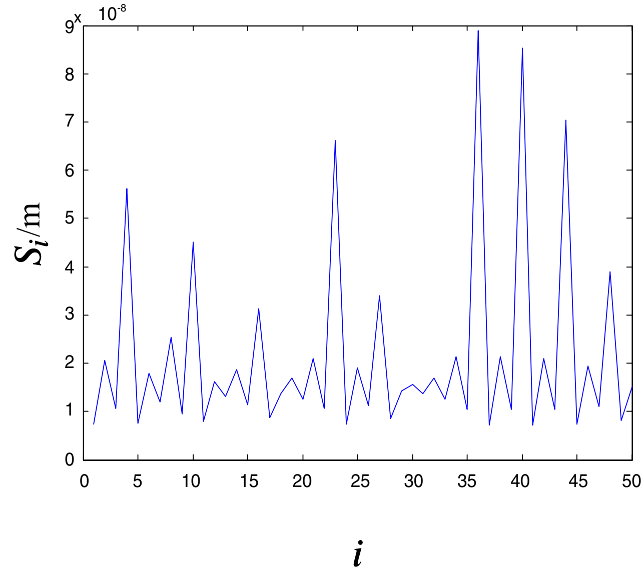


Fig. 6 Amplitude of $\{a_i,b_i\}$

confirmed by equation $\det(\mathfrak{A}) = 0$ and $\varsigma\Delta W = -2\xi\omega A_3 W$. The damping ratio ξ and frequency ω confirmed by the method mentioned above are content to the equation $\det(\mathfrak{A}) = 0$. Whereas, equation $\det(\mathfrak{A}) = 0$ only contains variable ξ and ω. The relation between the frequency ω and damping ratio can be discussed by it. As the above stator structure, Eq. 35 is used to ascertain the relation between the frequency of vibration and damping ratio. Figure 5 shows the relation between the frequency of vibration and damping ratio about B_{07}, B_{08} and B_{09} respectively.

Figure 5 expresses that when the damping ratio increases, the frequency of vibration will decrease.

5.2 The stator vibration

If a^* and b^* are presetted, a and b can be obtained by Eq. 38. Therefore, iterative method is used to solve Eq. 27 in the form of Eq. 36. Firstly, preset a_1 and b_1, then,

1. Let $a^*=a_i$, $b^*=b_i$;
2. Use Eq. 38 to obtain a and b; let $a_{i+1}=a$, $b_{i+1}=b$.

Repeat the process above, sequence $\{a_i,b_i\}$ can be obtained. If this sequence is convergent, the solution of Eq. 27 is $\{a,b\} = \lim_{i\to\infty}\{a_i,b_i\}$;

If the sequence is not convergent, let $\{a,b\} = \min_{a=a_i,b=b_i} \sqrt{(a_i - a_{i+1})^2 + (b_i - b_{i+1})^2}$

Note the amplitude of $\{a_i,b_i\}$ as $s_i = \sqrt{a_i^2 + b_i^2}$.

Table 3 shows the result of first ten steps of the iteration. It shows that, parameter $\{a_i,b_i\}$ changes in a fixed dimension and they can not expressed by a definite tendency. Figure 6 shows the tendency of $\{a_i,b_i\}$ changes by i. It shows that, $\{a_i,b_i\}$ is a periodic change as a whole. There are large skips in it.

6 Conclusion

An Electromechanical Coupling analyze model for the stator's vibration is established by Kirchhoff plate theory. The effect of shear deformation and rotary inertia is taken into account. The paper supposes that the main reason of the heat generated from the piezoelectric ceramic is its damping. The effect of the damping is considered in the model. The formulas of computing frequency of natural vibration and modal shape are given in the paper.

Table 3 Part of iterative value of $\{a_i,b_i\}$ unit μm.

i	1	2	3	4	5	6	7	8	9	10
a_i	−0.0769	−0.0070	0.01974	−0.0102	−0.0538	−0.0073	0.01714	−0.0114	−0.0243	−0.0091
b_i	0.0231	0.0020	−0.0056	0.0029	0.0160	0.0021	−0.0049	0.0033	0.0071	0.0026
s_I	0.0803	0.0072	0.0205	0.0106	0.0562	0.0076	0.0178	0.0119	0.0253	0.00940

Numerical simulation shows that the computed results basically accords to the results calculated by Finite Element method. It expresses the model proposed in this paper is correct, and it can be used to analyze the dynamics of ultrasonic motors. Numerical simulation also expresses that as the damping ratio increases, the vibration frequency will decrease.

The research also shows that, using the given method can obtain the solution of Eq. 27 in the form of Eq. 36 and estimate the value of the solution. The corresponding parameter $\{a_i, b_i\}$ is not convergent, but it changes periodically. It shows that, because of non-linear contact action, the vibration on the contact layer of motor is not a regular traveling wave. Its amplitude changes with time. This kind of fluctuation skips sometimes. The research shows that the amplitude of the vibration of the model's contact layer is about 10^{-8} m and the result is consistent with the result of the experiment. It validates that the model, the model can be used to analyze energy transform of ultrasonic motors.

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